

## ON SIMPLE SHAMSUDDIN DERIVATIONS IN TWO VARIABLES

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ABSTRACT. We study the subgroup of  $k$ -automorphisms of  $k[x, y]$  which commute with a simple derivation  $D$  of  $k[x, y]$ . We prove, for example, that this subgroup is trivial when  $D$  is a Shamsuddin simple derivation. In the general case of simple derivations, we obtain properties for the elements of this subgroup.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field of characteristic zero and the ring  $k[x, y]$  of polynomials over  $k$  in two variables.

A  $k$ -derivation  $d : k[x, y] \rightarrow k[x, y]$  of  $k[x, y]$  is a  $k$ -linear map such that

$$d(ab) = d(a)b + ad(b)$$

for any  $a, b \in k[x, y]$ . Denoting by  $\text{Der}_k(k[x, y])$  the set of all  $k$ -derivations of  $k[x, y]$ . Let  $d \in \text{Der}_k(k[x, y])$ . An ideal  $I$  of  $k[x, y]$  is called  $d$ -stable if  $d(I) \subset I$ . For example, the ideals  $0$  and  $k[x, y]$  are always  $d$ -stable. If  $k[x, y]$  has no other  $d$ -stable ideal it is called  $d$ -simple. Even in the case of two variables, a few examples of simple derivations are known (see for explanation [BLL2003], [Cec2012], [No2008], [BP2015], [KM2013] and [Leq2011]).

We denote by  $\text{Aut}(k[x, y])$  the group of  $k$ -automorphisms of  $k[x, y]$ . Let  $\text{Aut}(k[x, y])$  act on  $\text{Der}_k(k[x, y])$  by:

$$(\rho, D) \mapsto \rho^{-1} \circ D \circ \rho = \rho^{-1} D \rho.$$

Fixed a derivation  $d \in \text{Der}_k(k[x, y])$ . The isotropy subgroup, with respect to this group action, is

$$\text{Aut}(k[x, y])_D := \{\rho \in \text{Aut}(k[x, y]) / \rho D = D \rho\}.$$

We are interested in the following question proposed by I.Pan (see [B2014]):

**Conjecture 1.** *If  $d$  is a simple derivation of  $k[x, y]$ , then  $\text{Aut}(k[x, y])_d$  is finite.*

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At a first moment, in the §2, we show that the conjecture is true for a family of derivations, named Shamsuddin derivations (Theorem 6). For this, we use a theorem of the Shamsuddin [Sh1977], mentioned in [No1994, Theorem 13.2.1.], that determines a condition that would preserve the simplicity by extending, in some way, the derivation to  $R[t]$ , with  $t$  an indeterminate. The reader may also remember that Y. Lequain [Leq2011] showed that these derivations check a conjecture about the  $\mathbb{A}_n$ , the Weyl algebra over  $k$ .

In order to understand the isotropy of a simple derivation of the  $k[x, y]$ , in §3, we analysed necessary conditions for an automorphism to belong to the isotropy of a simple derivation. For example, we prove that if such an automorphism has a fixed point, then it is the identity (Proposition 7). Following, we present the definition of *dynamical degree* of a polynomial application and thus proved that in the case  $k = \mathbb{C}$ , the elements in  $\text{Aut}(\mathbb{C}[x, y])_d$ , with  $d$  a simple derivation, has dynamical degree 1 (Corollary 9). More precisely, the condition dynamical degree  $> 1$  corresponds to exponential growth of degree under iteration, and this may be viewed as a complexity of the automorphism in the isotropy (see [FM1989]).

## 2. SHAMSUDDIN DERIVATION

The main aim of this section is study the isotropy group of the a Shamsuddin derivation in  $k[x, y]$ . In [No1994, §13.3], there are numerous examples of these derivations and also shown a criterion for determining the simplicity; furthermore, Y. Lequain [Leq2008] introduced an algorithm for determining when an Shamsuddin derivation is simple. However, before this, the following example shows the isotropy of an arbitrary derivation can be complicated.

*Example 1.* Let be  $d = \partial_x \in \text{Der}_k(k[x, y])$  and  $\rho \in \text{Aut}(k[x, y])_d$ . Note that  $d$  is not a simple derivation; indeed, for any  $u(y) \in k[y]$ , the ideal generated by  $u(x)$  is always invariant. Consider

$$\begin{aligned}\rho(x) &= f(x, y) = a_0(x) + a_1(x)y + \dots + a_t(x)y^t \\ \rho(y) &= g(x, y) = b_0(x) + b_1(x)y + \dots + b_s(x)y^s.\end{aligned}$$

Since  $\rho \in \text{Aut}(k[x, y])_d$ , we obtain two conditions:

$$\mathbf{1)} \quad \rho(d(x)) = d(\rho(x)).$$

Thus,

$$1 = d(a_0(x) + a_1(x)y + \dots + a_t(x)y^t) = d(a_0(x)) + d(a_1(x))y + \dots + d(a_t(x))y^t.$$

Then,  $d(a_0(x)) = 1$  and  $d(a_j(x)) = 0$ ,  $j = 1, \dots, t$ . We conclude that  $\rho(x)$  is of the type

$$\rho(x) = x + c_0 + c_1y + \dots + c_ty^t, \quad c_i \in k.$$

$$\mathbf{2)} \quad \rho(d(y)) = d(\rho(y)).$$

Analogously,

$$0 = d(b_0(x) + b_1(x)y + \dots + b_s(x)y^s) = d(b_0(x)) + d(b_1(x))y + \dots + d(b_s(x))y^s.$$

That is,  $b_i(x) = d_i$  with  $d_i \in k$ . We conclude also that  $\rho(y)$  is of the type

$$\rho(y) = d_0 + d_1y + \dots + d_sy^s, \quad d_i \in k.$$

Thus,  $\text{Aut}(k[x, y])_d$  contains the affine automorphisms

$$(x + uy + r, uy + s),$$

with  $u, r, s \in k$ . In particular,  $\text{Aut}(k[x, y])_d$  is not finite.

Notice that  $\text{Aut}(k[x, y])_d$  contains also the automorphisms of the type  $(x + p(y), y)$ , with  $p(y) \in k[y]$ .

Now, we determine indeed the isotropy. Using only the conditions 1 and 2,

$$\rho = (x + p(y), q(y))$$

with  $p(y), q(y) \in k[y]$ . However,  $\rho$  is an automorphism, in other words, the determinant of the Jacobian matrix must be a nonzero constant. Thus,  $|J_\rho| = q'(y) = c$ ,  $c \in k^*$ . Therefore,  $\rho = (x + p(y), ay + c)$ , with  $p(y) \in k[y]$  and  $a, c \in k$ . Consequently,  $\text{Aut}(k[x, y])_d$  is not finite and, more than that, the first component has elements with any degree.

The following lemma is a well known result.

**Lemma 2.** *Let  $R$  be a commutative ring,  $d$  a derivation of  $R$  and  $h(t) \in R[t]$ , with  $t$  an indeterminate. Then, we can also extend  $d$  to a unique derivation  $\tilde{d}$  of the  $R[t]$  such that  $\tilde{d}(t) = h(t)$ .*

We will use the following result of Shamsuddin [Sh1977].

**Theorem 3.** *Let  $R$  be a ring containing  $\mathbb{Q}$  and let  $d$  be a simple derivation of  $R$ . Extend the derivation  $d$  to a derivation  $\tilde{d}$  of the polynomial ring  $R[t]$  by setting  $\tilde{d}(t) = at + b$  where  $a, b \in R$ . Then the following two conditions are equivalent:*

- (1)  $\tilde{d}$  is a simple derivation.
- (2) There exist no elements  $r \in R$  such that  $d(r) = ar + b$ .

*Proof.* See [No1994, Theorem 13.2.1.] for a demonstration in details. □

A derivation  $d$  of  $k[x, y]$  is said to be a *Shamsuddin derivation* if  $d$  is of the form

$$d = \partial_x + (a(x)y + b(x))\partial_y,$$

where  $a(x), b(x) \in k[x]$ .

*Example 4.* Let  $d$  be a derivation of  $k[x, y]$  as follows

$$d = \partial_x + (xy + 1)\partial_y.$$

Writing  $R = k[x]$ , we know that  $R$  is  $\partial_x$ -simple and, taking  $a = x$  and  $b = 1$ , we are exactly the conditions of Theorem 3. Thus, we know that  $d$  is simple if, and only if, there exist no elements  $r \in R$  such that  $\partial_x(r) = xr + 1$ ; but the right side of the equivalence is satisfied by the degree of  $r$ . Therefore, by Theorem 3,  $d$  is a simple derivation of  $k[x, y]$ .

**Lemma 5.** ([No1994, Proposition. 13.3.2]) *Let  $d = \partial_x + (a(x)y + b(x))\partial_y$  be a Shamsuddin derivation, where  $a(x), b(x) \in k[x]$ . Thus, if  $d$  is a simple derivation, then  $a(x) \neq 0$  and  $b(x) \neq 0$ .*

*Proof.* If  $b(x) = 0$ , then the ideal  $(y)$  is  $d$ -invariant. If  $a(x) = 0$ , let  $h(x) \in k[x]$  such that  $h' = b(x)$ , then the ideal  $(y - h)$  is  $d$ -invariant.  $\square$

One can determine the simplicity of the a Shamsuddin derivation according the polynomials  $a(x)$  and  $b(x)$  (see ([No1994, §13.3])).

**Theorem 6.** *Let  $D \in \text{Der}_k(k[x, y])$  be a Shamsuddin derivation. If  $D$  is a simple derivation, then  $\text{Aut}(k[x, y])_D = \{id\}$ .*

*Proof.* Let us denote  $\rho(x) = f(x, y)$  and  $\rho(y) = g(x, y)$ . Let  $D$  be a Shamsuddin derivation and

$$\begin{aligned} D(x) &= 1, \\ D(y) &= a(x)y + b(x), \end{aligned}$$

where  $a(x), b(x) \in k[x]$

Since  $\rho \in \text{Aut}(k[x, y])_D$ , we obtain two conditions:

- (1)  $\rho(D(x)) = D(\rho(x))$ .
- (2)  $\rho(D(y)) = D(\rho(y))$ .

Then, by condition (1),  $D(f(x, y)) = 1$  and since  $f(x, y)$  can be written in the form

$$f(x, y) = a_0(x) + a_1(x)y + \dots + a_s(x)y^s,$$

with  $s \geq 0$ , we obtain

$$\begin{aligned} D(a_0(x)) + D(a_1(x))y + a_1(x)(a(x)y + b(x)) + \dots \\ + D(a_s(x))y^s + sa_s(x)y^{s-1}(a(x)y + b(x)) = 1 \end{aligned}$$

Comparing the coefficients in  $y^s$ ,

$$D(a_s(x)) = -sa_s(x)a(x),$$

which can not occur by the simplicity. More explicitly, the Lemma 5 implies that  $a(x) = 0$ . Thus  $s = 0$ , this is  $f(x, y) = a_0(x)$ . Therefore  $D(a_0(x)) = 1$  and  $f = x + c$ , with  $c$  constant.

Using the condition (2),

$$\begin{aligned} D(g(x, y)) &= \rho(a(x)y + b(x)) \\ &= \rho(a(x))\rho(y) + \rho(b(x)) \\ &= a(x + c)g(x, y) + b(x + c) \end{aligned}$$

Writing  $g(x, y) = b_0(x) + b_1(x)y + \dots + b_t(x)y^t$ ; wherein, by the previous part, we can suppose that  $t > 0$ , because  $\rho$  is a automorphism. Thus

$$\begin{aligned} a(x + c)g(x, y) + b(x + c) &= D(b_0(x)) + D(b_1(x))y + b_1(x)(a(x)y + b(x)) + \\ &+ \dots + D(b_t(x))y^t + tb_t(x)y^{t-1}(a(x)y + b(x)). \end{aligned}$$

Comparing the coefficients in  $y^t$ , we obtain

$$D(b_t(x)) + tb_t(x)a(x) = a(x + c)b_t(x)$$

Then  $D(b_t(x)) = b_t(x)(-ta(x) + a(x + c))$ . In this way,  $b_t(x)$  is a constant and, consequently,  $a(x + c) = ta(x)$ . Comparing the coefficients in the last equality, we obtain  $t = 1$  and then  $b_1(x) = b_1$  a constant. Moreover, if  $a(x)$  is not a constant, since  $a(x + c) = a(x)$ , is easy to see that  $c = 0$ . Indeed, if  $c \neq 0$  we obtain that the polynomial  $a(x)$  has infinite distinct roots. If  $a(x)$  is a constant, then  $a(x)$   $D$  is not a simple derivation (a consequence of [Leq2008, Lemma.2.6 and Theorem.3.2]; thus, we obtain  $c = 0$ .

Note that  $g(x, y) = b_0(x) + b_1y$  and, using the condition (2) again,

$$\begin{aligned} D(g(x, y)) &= D(b_0(x)) + b_1(a(x)y + b(x)) \\ &= a(x)(b_0(x) + b_1y) + b(x). \end{aligned}$$

Considering the independent term of  $y$ ,

$$D(b_0(x)) = b_0(x)a(x) + b(x)(1 - b_1) \tag{1}$$

If  $b_1 \neq 1$ , we consider the derivation  $D'$  such that

$$D'(x) = 1, \quad D'(y) = a(x)y + b(x)(1 - b_1).$$

In [No1994, Proposition. 13.3.3], it is noted that  $D$  is a simple derivation if and only if  $D'$  is a simple derivation. Furthermore, by the Theorem 3, there exist no elements  $h(x)$  in  $K[x]$  such that

$$D(h(x)) = h(x)a(x) + b(x)(1 - b_1) :$$

what contradicts the equation (1). Then,  $b_1 = 1$  and  $D(b_0(x)) = b_0(x)a(x)$ , since  $D$  is a simple derivation we know that  $a(x) \neq 0$ , consequently  $b_0(x) = 0$ . This shows that  $\rho = id$ .  $\square$

### 3. ON THE ISOTROPY OF THE SIMPLE DERIVATIONS

The purpose of this section is to study the isotropy in the general case of a simple derivation. More precisely, we obtain results that reveal some characteristics of the elements in  $\text{Aut}(k[x, y])_D$ . For this, we use some concepts presented in the previous sections and also the concept of dynamical degree of a polynomial application.

In [BP2015], which was inspired by [BLL2003], we introduce and study a general notion of solution associated to a Noetherian differential  $k$ -algebra and its relationship with simplicity.

The following proposition geometrically says that if an element in the isotropy of a simple derivation has fixed point then it is the identity automorphism.

**Proposition 7.** *Let  $D \in \text{Der}_k(k[x_1, \dots, x_n])$  be a simple derivation and  $\rho \in \text{Aut}(k[x_1, \dots, x_n])_D$  an automorphism in the isotropy. Suppose that there exist a maximal ideal  $\mathfrak{m} \subset k[x_1, \dots, x_n]$  such that  $\rho(\mathfrak{m}) = \mathfrak{m}$ , then  $\rho = \text{id}$ .*

*Proof.* Let  $\varphi$  be a solution of  $D$  passing through  $\mathfrak{m}$  (see [BP2015, Definition.1.]). We know that  $\frac{\partial}{\partial t}\varphi = \varphi D$  and  $\varphi^{-1}((t)) = \mathfrak{m}$ . If  $\rho \in \text{Aut}(k[x_1, \dots, x_n])_D$ , then

$$\frac{\partial}{\partial t}\varphi\rho = \varphi D\rho = \varphi\rho D.$$

In other words,  $\varphi\rho$  is a solution of  $D$  passing through  $\rho^{-1}(\mathfrak{m}) = \mathfrak{m}$ . Therefore, by the uniqueness of the solution ([BP2015, Theorem.7.(c)]),  $\varphi\rho = \varphi$ . Note that  $\varphi$  is one to one, because  $k[x_1, \dots, x_n]$  is  $D$ -simple and  $\varphi$  is a nontrivial solution. Then, we obtain that  $\rho = \text{id}$ .

□

F. Lane, in [Lane75], proved that every  $k$ -automorphism  $\rho$  of  $k[x, y]$  leaves a nontrivial proper ideal  $I$  invariant, over an algebraically closed field; this is,  $\rho(I) \subseteq I$ . Em [Sh1982], A. Shamsuddin proved that this result does not extend to  $k[x, y, z]$ , proving that the  $k$ -automorphism given by  $\chi(x) = x + 1$ ,  $\chi(y) = y + xz + 1$  e  $\chi(z) = y + (x + 1)z$  has no nontrivial invariant ideal.

Note that, in addition,  $\rho$  leaves a nontrivial proper ideal  $I$  invariant if and only if  $\rho(I) = I$ , because  $k[x, y]$  is Noetherian. In fact, the ascending chain

$$I \subset \rho^{-1}(I) \subset \rho^{-2}(I) \subset \dots \subset \rho^{-l}(I) \subset \dots$$

must stabilize; thus, there exists a positive integer  $n$  such that  $\rho^{-n}(I) = \rho^{-n-1}(I)$ , then  $\rho(I) = I$ .

Suppose that  $\rho \in \text{Aut}(k[x, y])_D$  and  $D$  is a simple derivation of  $k[x, y]$ . If this invariant ideal  $I$  is maximal, by the Proposition 7, we have  $\rho = id$ .

Suppose that  $I$ , this invariant ideal, is radical. Let  $I = (\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s) \cap (\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t)$  be a primary decomposition where each ideal  $\mathfrak{m}_i$  is a maximal ideal and  $\mathfrak{p}_j$  are prime ideals with height 1 such that  $\mathfrak{p}_j = (f_j)$ , with  $f_j$  irreducible (by [Kaplan74, Theorem 5.]). If

$$\rho(\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s) = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s,$$

we claim that  $\rho^N$  leaves invariant one maximal ideal for some  $N \in \mathbb{N}$ : suppose  $\mathfrak{m}_1$  this ideal. Indeed, we know that  $\rho(\mathfrak{m}_1) \supset \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$ , since  $\rho(\mathfrak{m}_1)$  is a prime ideal, we deduce that  $\rho(\mathfrak{m}_1) \supseteq \mathfrak{m}_i$ , for some  $i = 1, \dots, s$  ([AM1969, Prop.11.1.(ii)]). Then,  $\rho(\mathfrak{m}_1) = \mathfrak{m}_i$ ; that is,  $\rho^N$  leaves invariant the maximal ideal  $\mathfrak{m}_1$  for some  $N \in \mathbb{N}$ . Thus follows from Proposition 7 that  $\rho^N = id$ .

Note that  $\rho(\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$ . In fact, writing  $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t = (f_1 \dots f_t)$ , with  $f_i$  irreducible, we can choose  $h \in \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$  such that  $\rho(h) \notin \mathfrak{p}_1$ . We observe that there exists  $h$ . Otherwise, we obtain  $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s \subset \mathfrak{p}_1$ , then  $\mathfrak{p}_1 \supseteq \mathfrak{m}_i$ , for some  $i = 1, \dots, s$  ([AM1969, Prop.11.1.(ii)]): a contradiction. Thus, since  $hf_1 \dots f_t \in I$ , we obtain  $\rho(h)\rho(f_1) \dots \rho(f_t) \in I \subset \mathfrak{p}_1$ . Therefore,  $\rho(f_1 \dots f_t) \in \mathfrak{p}_1$ . Likewise, we conclude the same for the other primes  $\mathfrak{p}_i$ ,  $i = 1, \dots, t$ . Finally,  $\rho(\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$ .

With the next corollary, we obtain some consequences on the last case.

**Corollary 8.** *Let  $\rho \in \text{Aut}(k[x, y])_D$ ,  $D$  a simple derivation of  $k[x, y]$  and  $I = (f)$ , with  $f$  reduced, a ideal with height 1 such that  $\rho(I) = I$ . If  $V(f)$  is singular or some irreducible component  $C_i$  of  $V(f)$  has genus greater than two, then  $\rho$  is a automorphism of finite order.*

*Proof.* Suppose that  $V(f)$  is not a smooth variety and let  $q$  be a singularity of  $V(f)$ . Since the set of the singular points is invariant by  $\rho$ , then there exist  $N \in \mathbb{N}$  such that  $\rho^N(q) = q$ . Using that  $\rho \in \text{Aut}(k[x, y])_D$ , we obtain, by Proposition 7,  $\rho^N = id$ .

Let  $C_i$  be a component irreducible of  $V(f)$  that has genus greater than two. Note that there exist  $M \in \mathbb{N}$  such that  $\rho^M(C_i) = C_i$ . By [FK1992, Thm. Hunvitz, p.241], the number of elements in  $\text{Aut}(C_i)$  is finite; in fact,  $\#(\text{Aut}(C_i)) < 84(g_i - 1)$ , where  $g_i$  is the genus of  $C_i$ . Then, we deduce that  $\rho$  is a automorphism of finite order.

□

We take for the rest of this section  $k = \mathbb{C}$ .

Consider a polynomial application  $f(x, y) = (f_1(x, y), f_2(x, y)) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  and define the degree of  $f$  by  $\deg(f) := \max(\deg(f_1), \deg(f_2))$ . Thus we may define the dynamical degree (see [BD2012], [FM1989], [Silv12]) of  $f$  as

$$\delta(f) := \lim_{n \rightarrow \infty} (\deg(f^n))^{\frac{1}{n}}.$$

**Corollary 9.** *If  $\rho \in \text{Aut}(\mathbb{C}[x, y])_D$  and  $D$  is a simple derivation of  $\mathbb{C}[x, y]$ , then  $\delta(\rho) = 1$ .*

*Proof.* Suppose  $\delta(\rho) > 1$ . By [FM1989, Theorem 3.1.],  $\rho^n$  has exactly  $\delta(\rho)^n$  fix points counted with multiplicities. Then, by Proposition 7,  $\rho = \text{id}$ , which shows that dynamical degree of  $\rho$  is 1.  $\square$

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